SIZES OF EXTENDED FORMULATIONS FOR FAMILIES OF POLYTOPES

Gennadiy Averkov OVGU Magdeburg, DE
Volker Kaibel OVGU Magdeburg, DE
Stefan Weltge ETH Zürich, CH

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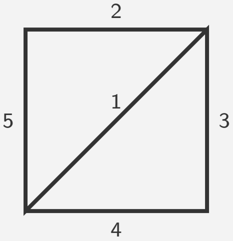
many prominent examples where this helps a lot ☺

Example: spanning tree problem

Graph with edges $1, \ldots, 5$



Spanning trees	0/1-Vectors
	$(1, 1, 1, 0, 0)$
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- On the other hand, the spanning tree polytope has a very small extended formulation.
- Thus, the spanning tree problem can be reduced to linear programming.
- There are many other examples of combinatorial problems, for which one can say: *it's just a special case of linear programming!*

Geometrically

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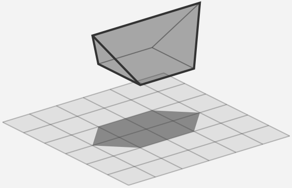

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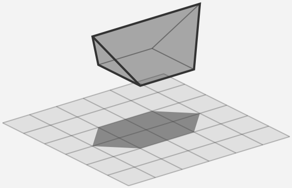

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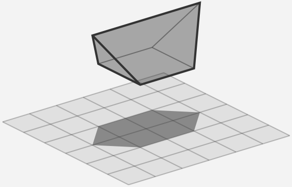

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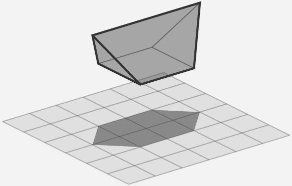

Using the cone $\mathbb{R}^k_+$

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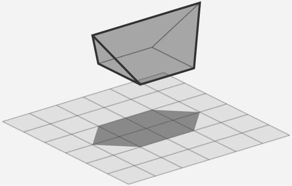

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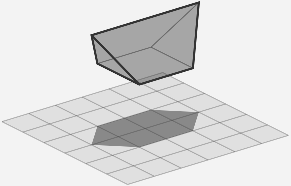

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Idea: Replace $\mathbb{R}^k_+$ by $\mathbb{S}^k_+$

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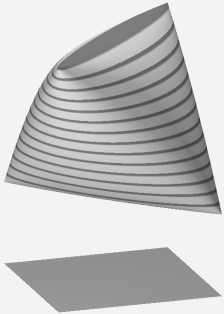

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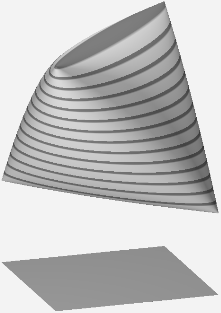

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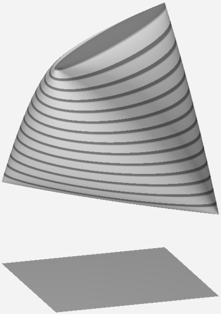

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The *semidefinite extension complexity* $\text{sxc}(P)$ of P is the smallest k such that $P \subseteq \mathbb{R}^d$ can be represented in the above way.

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It is an open question whether $\mathrm{xc}(P_{\mathrm{stab}}(G))$ is polynomial in $|V|$ if G is perfect.

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Application to 0/1-polytopes

defining $\mathcal{P}$ a bit more carefully one obtains:

Corollary

Let P be a random polytope uniformly distributed in the family of all polytopes with vertices in $\{0,1\}^d$. For d large enough we have

$$\mathrm{Prob}(\mathrm{sxc}(P) \leq 2^{0.24d}) \leq 2^{-2^{d-1}}.$$

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$\rightsquigarrow$ every normalized semidefinite ext. formulation is defined by a triple (A, φ, t)

Proof idea (2)

Parametrizations of normalized semidef. ef's

Bound parameters: $\|A\| \leq 1$, $\|\varphi\| \leq \rho$, $\|t\| \leq \rho$, $n \leq k^2$

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Bound distances: $\mathrm{dist}(P, P') \leq \rho n^2 \|A - A'\| + n\|\varphi - \varphi'\| + \|t - t'\|$

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Bound distances: $\mathrm{dist}(P, P') \leq \rho n^2 \|A - A'\| + n\|\varphi - \varphi'\| + \|t - t'\|$

Normed vector space of parametrizations (A, φ, t)

For every $w = (A, \varphi, t)$ define $\|w\| := \rho n^2 \|A\| + n\|\varphi\| + \|t\|$

Proof idea (2)

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Dimension of the vector space: $\leq 3dk^4$

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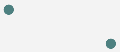

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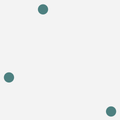

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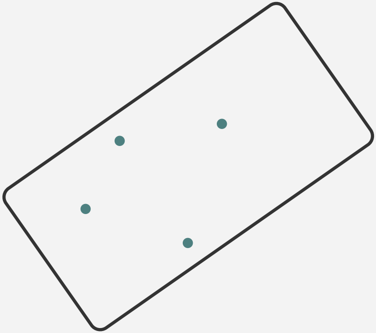

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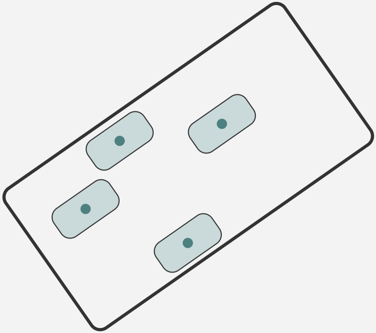

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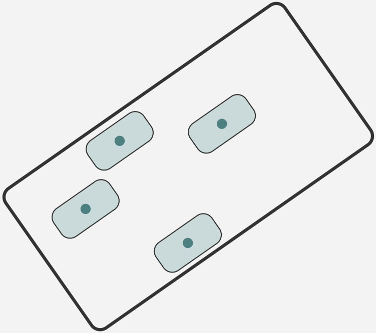

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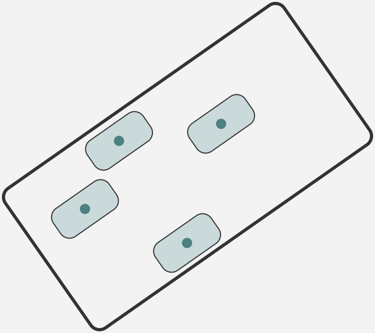

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$\rightsquigarrow |\mathcal{P}| \leq f(k, d, \rho, \Delta)$ $\square$

Extended formulations in combinatorial optimization

Known results

Main result

Application to 0/1-polytopes

Proof idea